Multi-dimensional Martingale Optimal Transport: Local structure and Numerics

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Outline

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Motivation: the model risk

- Major stake of last century for financial mathematics: pricing exotic derivatives.
- In practice the price is given by $\mathbb{E}^{\mathbb{P}}[\text{payoff}]$, with $\mathbb{P}$ martingale model fitting the prices of vanilla derivatives.
- Problem: two models $\mathbb{P}_1$ and $\mathbb{P}_2$ give two prices!
- Question: what is the maximal possible price?
- Martingale Optimal Transport problem: $\sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}[\text{payoff}]$.
- Understand the model risk and find riskless strategies.

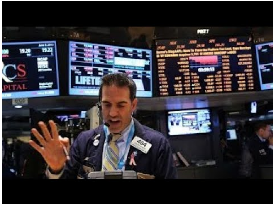

Figure: Jean-Michel, binary options trader.

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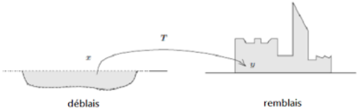
The Monge optimal transport problem

Originally a soil moving problem for building from Monge [22].

$$\inf_{\substack{T:\mathcal{X}\longrightarrow\mathcal{Y}\\ T\#\mu=\nu}} \int_{\mathcal{X}} |T(x)-x|\,\mu(dx).$$



(a) The Monge problem illustrated.

(b) The cost of moving bricks.

Figure: Mass moving and cost of moving it.

Probabilistic optimal transport

Kantorovitch [16] made this problem probabilistic.

- $\Omega = \mathbb{R}^d \times \mathbb{R}^d$
- X and Y the two canonical random variables $\Omega \to \mathbb{R}^d$, $X : (x, y) \mapsto x$ and $Y : (x, y) \mapsto y$.
- $\mathcal{P}(\mu, \nu) := \{\mathbb{P} \in \mathcal{P}(\Omega) : \mathbb{P} \circ X^{-1} = \mu, \mathbb{P} \circ Y^{-1} = \nu\}$, set of all coupling probability laws between μ and ν.

Definition

The optimal transport problem is:

$$\mathbf{P} = \inf_{\mathbb{P} \in \mathcal{P}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X, Y)]$$

Useful cost functions

- $(x, y) \mapsto d(x, y)$ where $d(\cdot, \cdot)$ is a distance.
- $(x, y) \mapsto |x - y|^2$
- $(x, y) \mapsto |x - y|$

Definition

We define the p-Wasserstein distance for $p \geq 1$: for $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$,

$$W^p(\mu, \nu) := \left(\inf_{\mathbb{P} \in \mathcal{P}(\mu, \nu)} \mathbb{E}^{\mathbb{P}}[|X - Y|^p] \right)^{\frac{1}{p}}$$

$(\mathcal{P}(\mathbb{R}^d), W^p)$ is a Polish space.

The dual problem

For $\varphi, \psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, let $\varphi \oplus \psi : (x, y) \longmapsto \varphi(x) + \psi(y)$.

Definition

The dual problem is

$$\mathbf{D} := \sup_{\varphi \oplus \psi \leq c} \mu[\varphi] + \nu[\psi]$$

Remark

If $\varphi \oplus \psi \leq c$ *and* $\mathbb{P} \in \mathcal{P}(\mu, \nu)$ *then*

$$\mu[\varphi] + \nu[\psi] = \mathbb{E}^{\mathbb{P}}[\varphi \oplus \psi] \leq \mathbb{E}^{\mathbb{P}}[c].$$

Therefore, $\mathbf{D} \leq \mathbf{P}$.

Kantorovitch Duality

The parallel study of these two problems is justified by the following theorem (see Villani 2008 [25])

Theorem

If the cost c is lower semicontinuous, then

- *Strong duality holds:* $\mathbf{P} = \mathbf{D}$.
- *There are optimizers* φ^*, ψ^* *for* $\mathbf{D}$ *and* $\mathbb{P}^* \in \mathcal{P}(\mu, \nu)$ *for* $\mathbf{P}$.

Proposition

$\mathbb{P} \in \mathcal{P}(\mu, \nu)$ *is concentrated on* $\Gamma := \{\varphi^*(X) + \psi^*(Y) = c(X, Y)\}$ *if and only if it is optimal.*

Structure of Optimal Transport

Theorem

If $y \mapsto \partial_x c(x, y)$ is injective,
then $\mathbb{P}^*[Y = T(X)] = 1$ with $T(x) = \partial_x c(x, \cdot)^{-1}(\nabla\varphi^*(x))$

Idea of the proof:

- $\Delta(x, y) := c(x, y) - \varphi^*(x) - \psi^*(y) \geq 0$,
- $\Delta(x, y) = 0$ if $(x, y) \in \Gamma = \operatorname{supp} \mathbb{P}^*$.
- Then $\partial_x \Delta(x, y) = 0$ if $(x, y) \in \Gamma$.
- $\partial_x c(x, y) = \nabla\varphi^*(x)$.

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Change in the Primal and the Dual Problems

Definition

The martingale optimal transport problem and its dual are:

$$\mathbf{P} = \sup_{\mathbb{P}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big],$$

and

$$\mathbf{D} := \inf_{\varphi\oplus\psi+h^{\otimes}\geq c} \mu[\varphi] + \nu[\psi].$$

With $\mathcal{M}(\mu,\nu) := \{\mathbb{P}\in\mathcal{P}(\mu,\nu) \text{ s.t. } \mathbb{E}^{\mathbb{P}}[Y|X] = X,\ a.s.\}$ and

$$h^{\otimes} := h(X)\cdot(Y-X), \quad \text{for} \quad h:\mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

Binary optimal models for specific cost functions

Beiglböck-Juillet [3], and Henry-Labordère-Touzi [15].

Theorem

We suppose that $d = 1$, $\mu << Leb$ and that $\partial_{xyy} c > 0$. Then

$$Card(\operatorname{supp} \mathbb{P}_X) \leq 2.$$

$$\Gamma = \left\{(x, T_d(x)), (x, T_u(x)) / x \in \mathbb{R}\right\}$$

Numerical solving possible.

Graphical interpretation

- $\partial_x c(x,y) = \nabla h(x) y + \nabla\varphi(x) - h(x) - \nabla h(x) x$, for $(x,y) \in \Gamma$.
- y in the intersection of an affine map with the graph of $\partial_x c(x,\cdot)$.
- Martingale Spence-Mirless condition: $\partial_{xyy} c > 0$, i.e. $\partial_x c$ strictly convex.

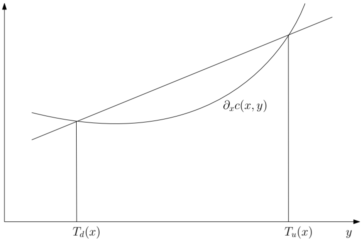

Figure: Structure of Γ_x.

The left-curtain coupling

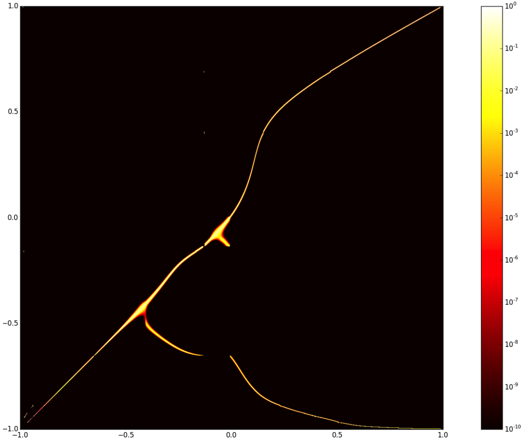

Figure: Left-curtain coupling.

Conjecture on higher dimension

- For $d = 1$, the conditional optimal transports are supported on $2 = d + 1$ points.
- As the optimal plans are extreme in $\mathcal{M}(\mu, \nu)$, we may conjecture that the optimal plans are supported on $d + 1$ points.
- $d + 1$ points convenient for using Partial Differential Equation methods.
- Ghoussoub-Kim-Lim [11] conjecture $d + 1$ points.
- Models for complete markets: $d + 1$ points.

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Structure theorem

DM 2018 [10].

Theorem

(i) *Under Assumption, we may find* $(A_x)_{x\in\mathbb{R}^d} \subset \mathrm{Aff}_d$ *such that for all* $\mathbb{P}^* \in \mathcal{M}(\mu,\nu)$ *optimal for MOT,*

$$\mathrm{supp}\ \mathbb{P}^*_x \subset \{\partial_x c(x,Y) = A_x(Y)\}, \text{ for } \mu - a.e.\ x \in \mathbb{R}^d.$$

(ii) *Conversely, let* $S_0 \subset \{\partial_x c(x_0,Y) = A(Y)\}$, *then under assumptions we may find* $\mu_0, \nu_0 \in \mathcal{P}(\mathbb{R}^d)$ *such that*

$$\mathbb{P}^*(dx,dy) := \mu_0(dx)\sum_{i=1}^k \lambda_i(x)\delta_{T_i(x)}(dy)$$

is the unique solution to MOT, and $S_0 = \{T_i(x_0) : 1 \le i \le k\}$.

Link with algebraic geometry

- Taylor: $\partial_x c(x_0, y) \approx \sum_{|j|\leq k} \frac{1}{|i|!}\partial_{x,y^j} c(x_0, y_0)(y - y_0)^j$.
- $\partial_x c(x_0, y) = A(y)$ is locally $P_1(y) = 0, ..., P_d(y) = 0$, with $P_i := \sum_{|j|\leq k} \frac{1}{|i|!}\partial_{x_i,y^j} c(x_0, y_0)(Y - y_0)^j - A_i(Y)$
- Algebraic geometry problem.
- No a priori information about A: $k \geq 2$, at infinity $P_i \approx \sum_{|j|=k} \frac{1}{|i|!}\partial_{x_i,y^j} c(x_0, y_0)(Y - y_0)^j$.
- Example: $\alpha Y^2 - A(Y)$ has 2 roots for all A if and only if $\alpha Y^2 \neq 0$.

Family of polynomial complete at infinity

Definition

We say that the family $(P_1, ..., P_d)$ is $\infty-$complete if

$$QP_i^{hom} \notin \langle P_1^{hom}, ..., P_{i-1}^{hom} \rangle, \text{ for all } Q \notin \langle P_1^{hom}, ..., P_{i-1}^{hom} \rangle.$$

where $< P_1^{hom}, ..., P_{i-1}^{hom} >$ is the ideal generated by the homogeneous part of $P_1, ..., P_{i-1}$.

Remark

Polynomial equation system $T \in \mathbb{R}\left[(X_{i,j})_{1\le i\le d, j\in(\mathbb{N}^*)^d:|j|\le k_i}\right]$ *such that for* $P_i = \sum_{j\in(\mathbb{N}^*)^d:|j|\le k_i} a_{i,j} X_1^{j_1}...X_d^{j_d}$, *we have*

$$T\left((a_{i,j})_{1\le i\le d, |j|\le k_i}\right) \neq 0 \iff (P_1, ..., P_d) \textit{ is } \infty-\textit{complete.}$$

Criterion of discreteness

Theorem (Bezout)

Let $d \in \mathbb{N}$ and $P_1, ..., P_d \in \mathbb{R}[X_1, ..., X_d]$ be $\infty-$complete. Then $|Z^{proj}(P_1, ..., P_d)| = deg(P_1)...deg(P_d)$, counted with multiplicity.

Theorem

If $\left(\sum_{\alpha\in\mathbb{N}^d,\,|\alpha|=k_i} \partial_{y^\alpha} c(x_0, y_0) Y^\alpha\right)_{1\leq i\leq d}$ is $\infty-$complete for all y_0, then $S_0 = \{\partial_x c(x_0, Y) = A(Y)\}$ is discrete.

Remark

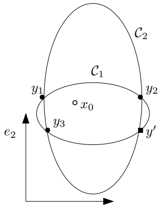
Warning: $c := f(|X - Y|)$ not second order $\infty-$complete.

An illustrative example

Example

Let $c : (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \longmapsto x_1(y_1^2 + 2y_2^2) + x_2(2y_1^2 + y_2^2)$. Then $\partial_x c(x, y) = (y_1^2 + 2y_2^2)e_1 + (2y_1^2 + y_2^2)e_2$. Let $A \in \mathrm{Aff}(\mathbb{R}^2, \mathbb{R}^2)$, $A = A_1 e_1 + A_2 e_2$. The equation $\partial_x c(x_0, y) = A(y)$ can be written

$$\begin{cases} y_1^2 + 2y_2^2 = & A_1(e_1)y_1 + A_1(e_2)y_2 + A_1(0) \\ 2y_1^2 + y_2^2 = & A_2(e_1)y_1 + A_2(e_2)y_2 + A_2(0). \end{cases}$$



Minimal number of mappings for smooth cost

Theorem

Let $c : \Omega \longrightarrow \mathbb{R}$ be second order complete at infinity and $\mathrm{C}^{2,0} \cap \mathrm{C}^{1,2}$ in the neighborhood of (x_0, x_0) for some $x_0 \in \mathbb{R}^d$. Then, we may find $\mu_0, \nu_0 \in \mathcal{P}(\mathbb{R}^d)$ such that the optimal coupling $\mathbb{P}^* \in \mathcal{M}(\mu_0, \nu_0)$ is unique and

$$|\mathrm{supp}\ \mathbb{P}^*_X| = N_c(x_0),\ \mu - a.s.$$

$$N_c(x_0) := \sup_{P \in \mathbb{R}_1[Y_1, \ldots, Y_d]^d} \left| Z^1_{\mathbb{R}}(H_c(x_0) + P) \right|,$$

where $H_c(x_0) := \sum_{i,j} \partial_{x, y_i y_j} c(x_0, x_0) Y_i Y_j$.

Value of $N_c(x_0)$

Theorem

Let $x_0 \in \mathbb{R}^d$, and c second order $\infty-$complete in (x_0, x_0) and $\mathrm{C}^{1,2}$ at (x_0, x_0), then

$$d + 1 + \mathbf{1}_{\{d\ even\}} \leq N_c(x_0) \leq 2^d.$$

Conjecture: $N_c(x_0) = 2^d$ for all second order complete at infinity c.

The case of the distance cost

Theorem

Let $c(X,Y) := |X-Y|^p$. Let $S_0 := \{\partial_x c(x_0,Y) = A(Y)\}$, for some $x_0 \in \operatorname{int}\operatorname{conv} S_0$, and $A \in \operatorname{Aff}(\mathbb{R}^d,\mathbb{R}^d)$.
(i) $p \leq 1$: S_0 contains $2d$ possibly degenerate points counted with multiplicity;
(ii) $p > 1$, S_0 contains $2d+1$ possibly degenerate points counted with multiplicity.

Degenerate points consist in $2k$ points with the same distance to x_0, then we replace these points with a $k-1$ dimensional sphere containing these points.

Limit: disintegration of the Lebesgue measure

- Assumption the the local structure result: μ_I dominated by Lebesgue.
- Invalid argument: the Sudakov mistake and the Nikodym set.
- Further work: prove that this cannot happen.

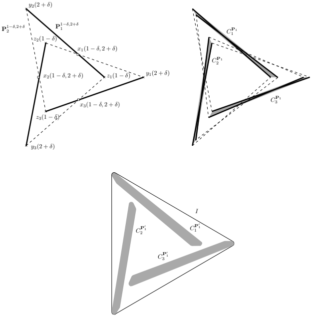


Figure: Three disjoint clusters in one single irreducible component.

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Existing methods in literature

- Linear programming: the Hungarian method [18], the auction algorithm [7], the network simplex [1], [12]...
- Polynomial time: only suited for small sized problems.
- Stochastic control: Benhamou & Brenier [4].
- Monge-Ampère: $\det D^2 u = \frac{g \circ c_x(X,\cdot)^{-1} \circ \nabla u}{f}$, where f is the density of μ and g is the density of ν: see [6] and [5].
- Semi-discrete approach Merigot [21], or Levy [20].
- Very fast but only works for specific (while relevant) costs.
- Entropic approach: Leonard [19], Marco Cuturi [8].

Numerical resolution: the entropic approach

- Adding regularity thanks to an entropic penalization.
- $\mathbf{P}^\varepsilon := \inf_{\mathbb{P}\in\mathcal{P}(\mu,\nu)} \mathbb{P}[c] + \varepsilon H(\mathbb{P})$,
  with $H(\mathbb{P}) := \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \big(\ln \mathbb{P}(x,y) - 1\big)\mathbb{P}(x,y)$.
- Gibbs shaped solution:
  $\mathbb{P} = \sum_{x,y} \exp\left(-\frac{c(x,y)-\varphi(x)-\psi(y)}{\varepsilon}\right)\delta_{(x,y)}$
- Dual problem:

$$\mathbf{D}^\varepsilon := \sup_{(\varphi,\psi)} \mu[\varphi]+\nu[\psi]-\varepsilon\sum_{x,y}\exp\left(-\frac{c(x,y)-\varphi(x)+\psi(y)}{\varepsilon}\right).$$

The Sinkhorn algorithm

- Smooth concave operator $V_\varepsilon(\varphi, \psi) := \mu[\varphi] + \nu[\psi] - \varepsilon \sum_{x,y} \exp\left(-\frac{c(x,y)-\varphi(x)-\psi(y)}{\varepsilon}\right)$.
- Euler-Lagrange equations $\partial_\varphi V_\varepsilon = 0$ (resp. $\partial_\psi V_\varepsilon = 0$) equivalent to the marginal relations $\mathbb{P} \circ X^{-1} = \mu$ (resp. $\mathbb{P} \circ Y^{-1} = \nu$).
- Marco Cuturi [8] gives the closed formulas: $\varphi(x) = -\varepsilon \ln\left(\frac{1}{\mu_x} \sum_y \exp\left(-\frac{c(x,y)-\psi(y)}{\varepsilon}\right)\right)$, and $\psi(y) = -\varepsilon \ln\left(\frac{1}{\nu_y} \sum_x \exp\left(-\frac{c(x,y)-\varphi(x)}{\varepsilon}\right)\right)$.
- Iterating these partial optimization: Sinkhorn algorithm [23], equivalent to a block optimization of V_ε.
- Converges exponentially fast, see Knight [17].

Existing litterature for numerical solving of MOT

- Henry-Labordère [14]: dual linear programming techniques for specific cost functions so that the dual constraints are much easier to check.
- Alfonsi, Corbetta & Jourdain [2]: difficulty to get a discrete approximation of continuous marginals in convex order, that are still in convex order in higher dimension.
- Guo & Oblój [13]: provide convergence results of the discrete problem to the continuous problem
- [13] provides an equivalent of the Sinkhorn algorithm for one dimensional MOT.
- Tan & Touzi [24]: used a dynamic programming approach to solve a continuous-time version of MOT.

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Entropic framework for MOT

DM 2018 [9].

$$\mathbf{P}^{\varepsilon} := \sup_{\mathbb{P}\in\mathcal{M}(\mu,\nu)}\mathbb{P}[c] - \varepsilon H(\mathbb{P})$$

$$= \mathbf{D}^{\varepsilon} := \inf_{(\varphi,\psi,h)} \mu[\varphi]+\nu[\psi]+\varepsilon\sum_{x,y}\exp\left(-\frac{\varphi\oplus\psi+h^{\otimes}-c}{\varepsilon}\right)(x,y).$$

- We denote $\Delta := \varphi\oplus\psi + h^{\otimes} - c$.
- The optimal coupling probability becomes $\mathbb{P} := \sum_{x,y}\exp\left(-\frac{\Delta(x,y)}{\varepsilon}\right)\delta_{(x,y)}$.
- the convex function to minimize becomes $V_{\varepsilon}(\varphi,\psi,h) := \mu[\varphi]+\nu[\psi]+\varepsilon\sum_{x,y}\exp\left(-\frac{\Delta(x,y)}{\varepsilon}\right)$.

The martingale Sinkhorn algorithm

- Sinkhorn algorithm completed by another step for the martingale relation: $0 = \frac{1}{\mu_x}\sum_y \exp\left(-\frac{\Delta(x,y)}{\varepsilon}\right)(y-x)$ $= \frac{\varepsilon}{\mu_x}\frac{\partial}{\partial h(x)}\sum_y \exp\left(-\frac{\Delta(x,y)}{\varepsilon}\right) = \frac{1}{\mu_x}\frac{\partial}{\partial h(x)}V_\varepsilon$.
- Martingale step not closed form: Newton algorithm for the smooth strongly convex function F_x of d variables given by

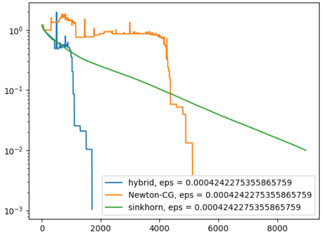
$$\begin{aligned}
F_x(h) &:= \frac{\varepsilon}{\mu_x}\sum_y \exp\left(-\frac{\Delta(x,y)}{\varepsilon}\right),\\
\nabla F_x(h) &= \frac{1}{\mu_x}\sum_y \exp\left(-\frac{\Delta(x,y)}{\varepsilon}\right)(y-x),\\
D^2F_x(h) &= \frac{1}{\mu_x\varepsilon}\sum_y \exp\left(-\frac{\Delta(x,y)}{\varepsilon}\right)(y-x)\otimes(y-x).
\end{aligned}$$

Implied conjugate gradient Newton algorithm

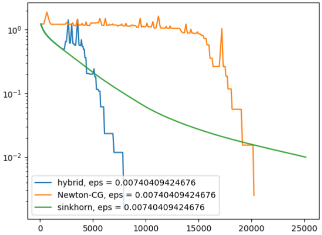
- The optimization in h is costly.
- Newton algorithm: $x_{n+1} - x_n = -D^2V(x_n)^{-1}\nabla V(x_n)$.
- Matrix inverting not adapted to high dimension.
- Conjugate gradient algorithm: find p such that $|D^2V(x_n)p - \nabla V(x_n)| \leq o(|\nabla V(x_n)|)$.
- Exploiting the closed formulas: if V_ε is $\alpha-$convex, so does $\widetilde{V}_\varepsilon(\psi) := \min_{\varphi,h} V_\varepsilon(\varphi,\psi,h)$.
- Optimization on $\widetilde{V}_\varepsilon$ by Newton conjugate gradient: much faster and much more stable.

Comparing the algorithms

- Figure 7a compares these algorithms, for a grid size 2500, with the cost function XY^2.
- Figure 7b compares then on a 160×160 grid, for $c : (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \longmapsto x_1(y_1^2 + 2y_2^2) + x_2(2y_1^2 + y_2^2)$.
- Newton efficient close to the optimum, and Sinkhorn gets close by closed formulas: we design an hybrid algorithm.

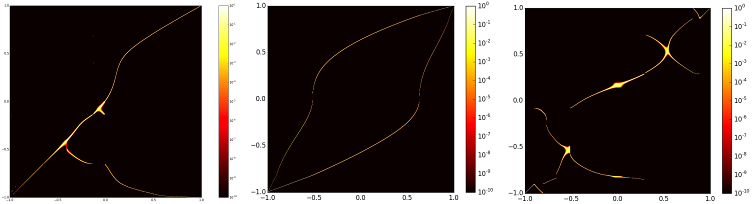


(a) Dimension 1.



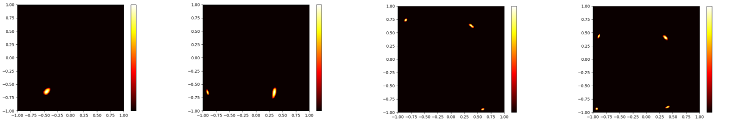
(b) Dimension 2.

The shapes of optimal transport



(a) $c = XY^2$. (b) $c = |X - Y|$. (c) $c = \sin(8XY)$.

Figure: Optimal coupling for different costs in dimension one.



(a) $x = -(0.45, 0.65)$ (b) $x = (0.3, -0.66)$ (c) $x = (0.2, 0.44)$ (d) $x = (0.13, 0.16)$

Default of convex order

- We consider the $\alpha-$penalized problem:

$$\min_{\psi\in\mathbb{R}^{\mathcal{Y}}} \widetilde{V}_\varepsilon(\psi) + \alpha f(\psi). \tag{6.1}$$

- f super-linear, strictly convex, homogeneous: enhances speed and stability.

Theorem

Let $(\mu,\nu)\in\mathcal{P}(\mathcal{X})\times\mathcal{P}(\mathcal{Y})$ not in convex order. Let $\nu_\alpha := \mathbb{P}_\alpha\circ Y^{-1}$, where $\mathbb{P}_\alpha$ is the optimal probability for Problem (6.1). Then $\nu_\alpha \longrightarrow \nu_l$ when $\alpha\longrightarrow 0$, for some $\nu_l \succeq_c \mu$ satisfying

$$f^*(\nu_l-\nu) = \min_{\tilde\nu\succeq_c\mu} f^*(\tilde\nu-\nu).$$

Entropy error

In the litterature the theoretical error linked to entropy is bad: $O(\varepsilon \log(\mathit{grid\ size}))$. Up to making uncheckable assumptions, we get a surprisingly universal and much better result.

Theorem

If

(i) $\Delta_\varepsilon := \varphi_\varepsilon \oplus \psi_\varepsilon + h_\varepsilon^\otimes - c$ *is* C^2;

(iii) $\mathrm{card} Z_x^\varepsilon = \dim Z_x^\varepsilon + 1$, *where* $Z_x^\varepsilon := \{\Delta_\varepsilon(x, Y) \approx 0\}$;

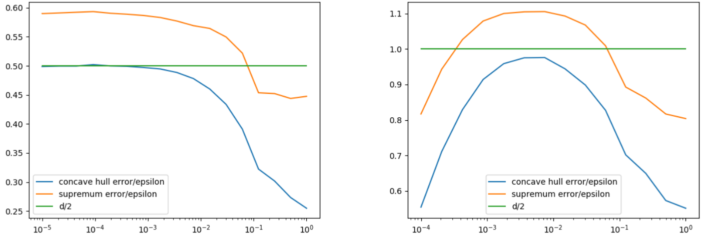
(vi) *the step of the* $Y-$*grid is* $o(\sqrt{\varepsilon})$.

Then $\frac{\mu_\varepsilon\left[(c(X,\cdot)-\psi_\varepsilon)_{conc}(X)\right]+\nu_\varepsilon[\psi_\varepsilon]-\mathbb{P}_\varepsilon[c]}{\varepsilon} \longrightarrow \frac{d}{2}$ *when* $\varepsilon \longrightarrow \infty$.

Corollary

Under the assumptions of Theorem 17, we have that $\mathbb{P}_\varepsilon[c] \geq \mathbf{S}_{\mu,\nu}(c) - \frac{d}{2}\varepsilon + o(\varepsilon)$, *when* $\varepsilon \longrightarrow 0$.

Numerical entropy error



(a) Dimension 1.

(b) Dimension 2.

Figure: Duality gap for the supremum, and the concave hull dual approximation vs ε.

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Questions?

Figure: Optimal transport in practice.